

INDESTRUCTIBILITY OF VOPĚNKA'S PRINCIPLE

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Abstract. We show that Vopěnka's Principle and Vopěnka cardinals are indestructible under reverse Easton forcing iterations of increasingly directed-closed partial orders, without the need for any preparatory forcing. As a consequence, we are able to prove the relative consistency of these large cardinal axioms with a variety of statements known to be independent of ZFC, such as the generalised continuum hypothesis, the existence of a definable well-order of the universe, and the existence of morasses at many cardinals.

§1. Introduction. Vopěnka's Principle is a large cardinal axiom that can readily be expressed in either set- or category-theoretic terms. It has received significant attention in the latter context, yielding structural results for certain important kinds of categories, as described in the final chapter of Adámek and Rosický's book [1]. This has led to the resolution under the assumption of Vopěnka's Principle of a long-standing open question in algebraic topology: Casacuberta, Scevenels and Smith [10] have shown that if Vopěnka's Principle holds, then Bousfield localisation functors exist for all generalised cohomology theories.

From a set-theoretic perspective, whilst it has received some attention (for example in [6], [22], [28], [30], and [31]), Vopěnka's Principle has generally been overshadowed by other large cardinal axioms, and there are many natural questions regarding it that remain unanswered. One aim of this article is to remedy the situation somewhat, providing the means for obtaining relative consistency results for Vopěnka's Principle with various other statements known to be independent of ZFC. Since Vopěnka's Principle lies beyond the scope of current inner model theory, this entails an analysis of the interaction between Vopěnka's Principle and the other standard technique for obtaining consistency results, namely, Cohen's method of forcing.

Specifically, we consider the common approach to obtaining consistency results for very large cardinals by forcing, whereby one starts with a model of ZFC containing the desired large cardinals, and forces other statements to hold, whilst preserving the large cardinal of interest. In many cases this may be achieved by Silver's technique of lifting embeddings, with a generic chosen to contain a particular "master condition" that forces the large cardinal to be preserved (see for example Cummings [11]). Laver [25] showed that with a suitable

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preparatory forcing, a supercompact cardinal κ can be made indestructible under κ -directed closed forcing, in that *any* generic for such a forcing will preserve the supercompactness of κ . Since then a variety of such indestructibility results have been obtained — see for example [2], [3], [4], [16], [17], [18] and [21]. In each case, a preparatory forcing is used to ensure that the large cardinal is (or cardinals are) indestructible in the extension universe.

We shall show here that Vopěnka’s Principle is in fact always indestructible under a useful class of forcings, with no preparation necessary. Specifically, we show that reverse Easton iterations of increasingly directed-closed forcings preserve Vopěnka’s Principle. As a warm-up, we first show that Vopěnka’s Principle is indestructible under small forcing, analogously to the Lévy–Solovay Theorem for measurable cardinals. Key to both arguments is the fact that Vopěnka’s Principle is witnessed by many embeddings, so it is not important to lift any particular one. It suffices to lift *some* embedding for each proper class of structures, and this is forced by a dense set of conditions in each case.

We have just alluded to the fact that the naïve statement of Vopěnka’s Principle involves proper classes. A class theory such as von Neumann–Bernays–Gödel or Morse–Kelley might consequently seem like the “right” context in which to study Vopěnka’s Principle, particularly as we shall have to be careful about use of the Global Axiom of Choice. However, we accede to modern set-theoretic tastes, and work within Zermelo–Fraenkel set theory. There are two approaches to this. One is to take “classes” to mean definable classes, and consider Vopěnka’s Principle to be an axiom schema, as for example in the recent work of Bagaria [6]. Alternatively, one can consider inaccessible cardinals κ such that V_κ satisfies a class theoretic form of Vopěnka’s Principle when all members of $V_{\kappa+1}$ are taken to be classes, as in Kanamori [23]. We shall first prove our indestructibility result for such *Vopěnka cardinals*, in order that the main ideas might not be obscured by the technicalities involved with dealing with definable proper classes. That done, we shall in Section 6 address those technicalities and how they may be overcome to yield the corresponding results for the definable class version of Vopěnka’s Principle.

§2. Definitions. In order to define Vopěnka’s Principle, we fix some model-theoretic notation. It will be convenient to refer to a standard language \mathcal{L}_{std} with one binary relation symbol ϵ and one unary relation symbol R . Since any number of relations of any arity can be encoded in a single binary relation (see for example Pultr and Trnková [29, Theorem II.5.3]), this is tantamount to considering all languages. Unless otherwise specified, M will denote the domain of any model denoted \mathcal{M} , and N will denote the domain of any model denoted \mathcal{N} .

DEFINITION 1. Let κ be an inaccessible cardinal and let A be a subset of V_κ of cardinality κ such that every element of A is an \mathcal{L}_{std} -structure. We denote by $VP(A)$ the statement that there are distinct \mathcal{M} and \mathcal{N} in A such that there exists an \mathcal{L}_{std} -elementary embedding $j : \mathcal{M} \rightarrow \mathcal{N}$.

DEFINITION 2. A cardinal κ is a Vopěnka cardinal if κ is inaccessible, and for every $A \subset V_\kappa$ of cardinality κ such that every element of A is an \mathcal{L}_{std} -structure, $VP(A)$ holds.

DEFINITION 3. Vopěnka's Principle is the axiom schema that states the following.

VP: For every (definable) proper class of \mathcal{L}_{std} -structures, there are distinct members \mathcal{M} and \mathcal{N} of the class such that there is an \mathcal{L}_{std} -elementary embedding $j : \mathcal{M} \rightarrow \mathcal{N}$.

An alternative way to define Vopěnka cardinals and Vopěnka's Principle is to remove the above distinctness requirements on \mathcal{M} and \mathcal{N} , and only require that the embedding j be non-trivial; we have chosen to follow Solovay, Reinhardt and Kanamori [31] in this regard. Since rigid graphs can be constructed of any cardinality using the axiom of choice (see [34]), and structures with two binary relations can be encoded into graphs in a way that respects homomorphisms (see for example [29]), these formulations are equivalent under the assumption of global choice. In the definable class setting, global choice is equivalent to $V = \text{HOD}$. We shall avoid using this assumption, and show in Section 6 that in fact $V = \text{HOD}$ may be forced while preserving Vopěnka's Principle (but note that the form of Vopěnka's Principle with which we work is the *stronger* version of the two alternatives in the absence of $V = \text{HOD}$).

We now focus on Vopěnka cardinals, leaving the details of the corresponding results for Vopěnka's Principle to Section 6. It will be convenient to have at our disposal another equivalent but more restricted characterisation of Vopěnka cardinals

DEFINITION 4. For any language \mathcal{L} , an ordinal \mathcal{L} -structure is an \mathcal{L} -structure with domain an ordinal.

LEMMA 5. For any inaccessible cardinal κ , κ is Vopěnka if and only if for every set $B \subset V_\kappa$ of cardinality κ of ordinal \mathcal{L}_{std} -structures, there exist distinct \mathcal{M} and \mathcal{N} in B such that there is an elementary embedding $j : \mathcal{M} \rightarrow \mathcal{N}$.

PROOF. Any A as in Definition 2 may clearly be converted to a corresponding set B of ordinal \mathcal{L}_{std} -structures using a choice function to choose for each element \mathcal{M} of A a unique representative from the set of ordinal \mathcal{L}_{std} -structures isomorphic to \mathcal{M} . An elementary embedding between distinct members of B then corresponds to an elementary embedding between distinct members of A . \dashv

Note that a naïve recasting of this proof in terms of definable classes would require $V = \text{HOD}$. However, we shall see in Section 6 that this is not necessary for the corresponding lemma to hold.

We will also need the following characterisation of Vopěnka cardinals, which is a slight refinement of one from Kanamori [23].

DEFINITION 6. *Let A be a set and η an ordinal less than κ . A cardinal $\alpha < \eta$ is η -extendible below κ for A if there is some $\zeta < \kappa$ and an elementary embedding $j : \langle V_\eta, \in, A \cap V_\eta \rangle \rightarrow \langle V_\zeta, \in, A \cap V_\zeta \rangle$ with critical point α and $j(\alpha) > \eta$. A cardinal $\alpha < \kappa$ is extendible below κ for A if it is η -extendible below κ for A for all η strictly between α and κ .*

PROPOSITION 7. *The following are equivalent for inaccessible cardinals κ :*

1. κ is a Vopěnka cardinal,
2. for every $A \subset V_\kappa$, there is a cardinal $\alpha < \kappa$ that is extendible below κ for A .

PROOF. The proof of Exercise 24.19 of Kanamori [23] also proves this “below κ ” refinement. \dashv

We mostly follow the notational conventions of Kunen [24] for forcing concepts; in particular, $q \leq p$ shall mean that q is a stronger condition than p , and for any set x in the ground model, we shall denote by \check{x} the canonical name for x in the extension, $\{\langle \check{y}, \mathbb{1} \rangle \mid y \in x\}$.

§3. κ^+ -distributive forcing and \square . The following Proposition is immediate from the definition of Vopěnka cardinals.

PROPOSITION 8. *If κ is a Vopěnka cardinal and \mathbb{P} is a κ^+ -distributive partial order (that is, forcing with \mathbb{P} adds no new subsets of κ), then κ remains a Vopěnka cardinal after forcing with \mathbb{P} .* \dashv

Whilst Proposition 8 is entirely trivial, it can be used to make the following interesting observation. Recall that for any uncountable cardinal κ , a \square_κ -sequence is a sequence $\langle C_\alpha \mid \alpha \in \lim \cap \kappa^+ \rangle$ such that for all $\alpha \in \lim \cap \kappa^+$,

- C_α is a club in α ,
- $\text{ot}(C_\alpha) \leq \kappa$,
- if $\beta \in \lim(C_\alpha)$ then $C_\beta = C_\alpha \cap \beta$.

The statement that there exists a \square_κ -sequence is denoted simply by \square_κ .

COROLLARY 9. *It is relatively consistent with “ κ is a Vopěnka cardinal” that \square_κ holds.*

PROOF. The usual forcing (due to Jensen) to make \square_κ hold, in which the conditions are partial \square_κ sequences, is $< \kappa^+$ -strategically closed (see for example Cummings [11, Sections 5 and 6]), and in particular is κ^+ -distributive. \dashv

This contrasts with the result of Jensen (see Friedman [15]) that if κ is subcompact then \square_κ fails: subcompact cardinals are consistency-wise weaker than Vopěnka cardinals. For further discussion of \square_κ and its failure for subcompact and related cardinals κ , see Cummings and Schimmerling [13, Section 6].

Proposition 8 will also be important for showing that Vopěnka cardinals κ are preserved by appropriate forcing iterations that go beyond κ .

COROLLARY 10. *If κ is a cardinal and $\mathbb{P} * \dot{\mathbb{Q}}$ is a forcing iteration such that*

$$\Vdash_{\mathbb{P}} \check{\kappa} \text{ is a Vopěnka cardinal} \wedge \dot{\mathbb{Q}} \text{ is } \check{\kappa}^+ \text{-distributive}$$

*then $\Vdash_{\mathbb{P} * \dot{\mathbb{Q}}} \check{\kappa} \text{ is a Vopěnka cardinal}$.*

⊢

§4. Small forcing. It has become part of the set-theoretic folklore that small forcing preserves most large cardinals, where by “small” we mean of cardinality less than the large cardinal in question. This stems from the original result of Lévy and Solovay [26] that small forcing preserves measurable cardinals, and the fact that most strong large cardinal properties can be expressed similarly to measurable cardinals with a witnessing elementary embedding. Whilst the definition of Vopěnka cardinals does involve elementary embeddings, it is not immediately clear that the usual argument will extend to this case. In this section we shall confirm that these large cardinal properties are preserved by small forcing, and in the process set the scene for later sections. Of course, this could be considered to be a special case of our main theorem, but there are tricks we can use to simplify the argument significantly in this context for the reader not interested in the full reverse Easton iteration result.

We need the following well-known, basic large cardinal preservation result.

LEMMA 11. *If κ is an inaccessible cardinal in V , \mathbb{P} is a partial order of cardinality less than κ , and G is \mathbb{P} -generic over V , then κ is inaccessible in $V[G]$.*

PROOF. For any partial order \mathbb{P} , \mathbb{P} is $|\mathbb{P}|^+$ -cc, and hence preserves cofinalities greater than $|\mathbb{P}|$ and the continuum function $\lambda \mapsto 2^\lambda$ for $\lambda \geq |\mathbb{P}|$. ⊢

A key point in the proof of Theorem 14, and indeed our later theorems, is that the embeddings in which we are interested for Vopěnka cardinals need not respect the given subset A of V_κ , but rather be elementary between two elements of A ; and yet no particular element is especially important, as there will be many embeddings witnessing Vopěnka’s Principle for each subset A of V_κ . This means that we can replace a name \dot{A} for a subset of V_κ by one consisting of particularly nice names for its elements, giving us much more control.

DEFINITION 12. Let \mathbb{P} be a forcing partial order and \mathcal{L} a relational language. A nice \mathbb{P} -name for an ordinal \mathcal{L} -structure is a \mathbb{P} -name σ of the following form:

1. σ is the canonical name for an ordinal \mathcal{L} -structure $\langle \gamma_\sigma, R^\sigma \mid R \in \mathcal{L} \rangle$ in $V[G]$ with components named by $\check{\gamma}_\sigma$ and \dot{R}^σ for $R \in \mathcal{L}$,
2. if $R \in \mathcal{L}$ is n -ary, the name \dot{R}^σ for R^σ is of the form

$$\bigcup_{(\beta_1, \dots, \beta_n) \in \gamma_\sigma^n} \{(\beta_1, \dots, \beta_n)\} \times A_{(\beta_1, \dots, \beta_n)}^R$$

where $A_{(\beta_1, \dots, \beta_n)}^R$ is an antichain in \mathbb{P} for each $(\beta_1, \dots, \beta_n) \in \gamma_\sigma^n$.

LEMMA 13. Let \mathcal{L} be a language, \mathbb{P} a partially ordered set, \dot{A} a \mathbb{P} -name, and p an element of \mathbb{P} such that

$$p \Vdash \forall a \in \dot{A} (a \text{ is an ordinal } \mathcal{L}\text{-structure}).$$

Then there is a \mathbb{P} -name \dot{B} such that $p \Vdash \dot{A} = \dot{B}$ and for every element $\langle \sigma, q \rangle$ of \dot{B} , $q \leq p$ and σ is a nice \mathbb{P} -name for an ordinal \mathcal{L}_{std} -structure.

PROOF. The argument is a fairly typical “nice-name” construction. For each $\langle \dot{a}, r \rangle \in \dot{A}$, let

$$Q_{\dot{a}, r} = \{q \in \mathbb{P} \mid q \leq p \wedge q \leq r \wedge \exists \gamma (q \Vdash \text{dom}(\dot{a}) = \gamma)\}.$$

Suppose we have $q \in Q_{\dot{a}, r}$ and γ such that $q \Vdash \text{dom}(\dot{a}) = \gamma$. For each n -ary relation symbol $R \in \mathcal{L}$ and each $(\beta_1, \dots, \beta_n) \in \gamma^n$, we can choose an antichain $A_{(\beta_1, \dots, \beta_n)}$ below q such that for each $s \in A_{(\beta_1, \dots, \beta_n)}$ we have $s \Vdash (\beta_1, \dots, \beta_n) \in R^{\dot{a}}$, and such that $A_{(\beta_1, \dots, \beta_n)}$ is maximal with this property (here $R^{\dot{a}}$ denotes the canonical name for the interpretation of the relation R in the structure named by \dot{a}). Then setting $\dot{R}_{(\beta_1, \dots, \beta_n)}^{q, \dot{a}} = \{(\beta_1, \dots, \beta_n)\} \times A_{(\beta_1, \dots, \beta_n)}$, we have by standard arguments that

$$q \Vdash (\beta_1, \dots, \beta_n) \in R^{\dot{a}} \leftrightarrow (\beta_1, \dots, \beta_n) \in \dot{R}_{(\beta_1, \dots, \beta_n)}^{q, \dot{a}}.$$

Taking $\dot{R}^{\sigma_{\dot{a}, r, q}} = \bigcup_{(\beta_1, \dots, \beta_n) \in \gamma^n} \dot{R}_{(\beta_1, \dots, \beta_n)}^{q, \dot{a}}$, and $\gamma_{\sigma_{\dot{a}, r, q}} = \gamma$, we obtain a nice \mathbb{P} -name $\sigma_{\dot{a}, r, q}$ for an ordinal \mathcal{L} -structure such that $q \Vdash \sigma_{\dot{a}, r, q} = \dot{a}$. Taking $\dot{B} = \bigcup_{\langle \dot{a}, r \rangle \in \dot{A}} \{\langle \sigma_{\dot{a}, r, q}, q \rangle \mid q \in Q_{\dot{a}, r}\}$, we have $p \Vdash \dot{A} = \dot{B}$, as required. \dashv

THEOREM 14. Suppose κ is a Vopěnka cardinal in V , \mathbb{P} is a partially ordered set of cardinality less than κ , and G is \mathbb{P} -generic over V . Then κ is Vopěnka in $V[G]$.

PROOF. By replacing \mathbb{P} with an isomorphic partial order if necessary, we may assume for convenience that the underlying set of \mathbb{P} is the cardinal $|\mathbb{P}| < \kappa$. We know from Lemma 11 that κ is inaccessible in $V[G]$, so it suffices to show that for any set A of cardinality κ of ordinal \mathcal{L}_{std} -structures in $(V_\kappa)^{V[G]}$, there are distinct elements \mathcal{M}

and \mathcal{N} of A with an elementary embedding $j : \mathcal{M} \rightarrow \mathcal{N}$ in $V[G]$. Let \dot{A} be a \mathbb{P} -name for A , and let $p \in G$ be such that

$$p \Vdash (\dot{A} \subset V_{\check{\kappa}}) \wedge (|\dot{A}| = \check{\kappa}) \wedge \forall a \in \dot{A} (a \text{ is an ordinal } \mathcal{L}_{\text{std}}\text{-structure}).$$

We shall show that $p \Vdash \text{VP}(\dot{A})$.

Our approach will be to show that it is dense below p to force $\text{VP}(\dot{A})$, for then p will also force $\text{VP}(\dot{A})$. So suppose we have $r \leq p$; by Lemma 13 there is a name \dot{B}_r such that $r \Vdash \dot{A} = \dot{B}_r$, and for every element $\langle \sigma, q \rangle$ of \dot{B}_r , $q \leq r$ and σ is a nice \mathbb{P} -name for an ordinal \mathcal{L}_{std} -structure. To avoid concerns about the distinctness of the \mathcal{M} and \mathcal{N} we find, we may thin out \dot{B}_r to a name \dot{C} , still with $|\dot{C}| = \kappa$, such that if $\langle \sigma_0, q_0 \rangle \neq \langle \sigma_1, q_1 \rangle$ are both in \dot{C} , then $\gamma_{\sigma_0} \neq \gamma_{\sigma_1}$, and so certainly $\sigma_0^{G'} \neq \sigma_1^{G'}$ in any generic extension by a \mathbb{P} -generic G' containing r .

Let $R_{\mathbb{P}}$ be a rigid binary relation on $|\mathbb{P}|$, that is, one admitting no non-identity endomorphism; see [1], [29], or the original paper of Vopěnka, Pultr and Hedrlín [34] for such a construction. Now in V consider the set

$$D = \left\{ \langle H_{\max(|\text{trcl}(\sigma)|, |\mathbb{P}|)^+}, \in, \langle \sigma, q, R_{\mathbb{P}} \rangle \rangle \mid \langle \sigma, q \rangle \in \dot{C} \right\}.$$

Since κ is a Vopěnka cardinal in V , there are $\mathcal{M} \neq \mathcal{N}$ in D with an elementary embedding $j : \mathcal{M} \rightarrow \mathcal{N}$. Suppose

$$\mathcal{M} = \langle H_{\alpha}, \in, \langle \sigma_{\mathcal{M}}, q_{\mathcal{M}}, R_{\mathbb{P}} \rangle \rangle$$

and

$$\mathcal{N} = \langle H_{\beta}, \in, \langle \sigma_{\mathcal{N}}, q_{\mathcal{N}}, R_{\mathbb{P}} \rangle \rangle;$$

we shall show that in any generic extension $V[G']$ by a \mathbb{P} -generic G' containing r , $j \upharpoonright \gamma_{\sigma_{\mathcal{M}}} : \gamma_{\sigma_{\mathcal{M}}} \rightarrow \gamma_{\sigma_{\mathcal{N}}}$ is elementary when considered as a map $\sigma_{\mathcal{M}}^{G'} \rightarrow \sigma_{\mathcal{N}}^{G'}$.

For any \mathcal{L}_{std} -formula φ , $\sigma_{\mathcal{M}}^{G'} \models \varphi(\alpha_1, \dots, \alpha_n)$ if and only if there is some $q \in G'$ such that

$$(*) \quad q \Vdash (\sigma_{\mathcal{M}} \models \varphi(\check{\alpha}_1, \dots, \check{\alpha}_n)).$$

Since satisfaction (for set models) is Σ_1 -definable, the statement $(*)$ is also Σ_1 for models containing \mathbb{P} ; indeed, it can be written in the form

$$\exists \dot{s} \exists F \exists X ($$

F is the characteristic function of the \Vdash relation

for Δ_0 formulae on the (sufficiently large) transitive set X ,

and F witnesses that

$q \Vdash$ “ \dot{s} is the characteristic function of the relation $\sigma_{\mathcal{M}} \models$ ”,

and F witnesses that

$q \Vdash$ “ \dot{s} witnesses that $\sigma_{\mathcal{M}} \models \varphi(\check{\alpha}_1, \dots, \check{\alpha}_n)$ ”)

where the main parenthesised part is Δ_0 . As originally shown by Lévy [27], one can prove with a Löwenheim-Skolem argument that H_λ is Σ_1 -elementary in V for any uncountable cardinal λ . In particular, H_α , and hence \mathcal{M} , is correct for the statement (*). By elementarity and the Σ_1 -correctness of \mathcal{N} , we thus have

$$q \Vdash (\sigma_{\mathcal{M}} \models \varphi(\check{\alpha}_1, \dots, \check{\alpha}_n)) \leftrightarrow j(q) \Vdash (\sigma_{\mathcal{N}} \models \varphi(j(\check{\alpha}_1), \dots, j(\check{\alpha}_n))).$$

By the rigidity of $R_{\mathbb{P}}$, $j(q) = q$ for any $q \in \mathbb{P}$, and we may conclude that for any \mathbb{P} -generic G' containing r , $\sigma_{\mathcal{M}}^{G'} \models \varphi(\alpha_1, \dots, \alpha_n)$ if and only if $\sigma_{\mathcal{N}}^{G'} \models \varphi(j(\alpha_1), \dots, j(\alpha_n))$.

Finally, $q_{\mathcal{M}} = q_{\mathcal{N}}$ by rigidity once more, and if $q_{\mathcal{M}} \in G'$, then $\sigma_{\mathcal{M}}^{G'}$ and $\sigma_{\mathcal{N}}^{G'}$ are elements of $\dot{C}^{G'} \subset (\dot{B}_r)^{G'} = \dot{A}^{G'}$. Hence, $q_{\mathcal{M}} \Vdash \text{VP}(\dot{A})$. But now $q_{\mathcal{M}} \leq r$ by the construction of \dot{B}_r . Therefore, it is dense below p to force $\text{VP}(\dot{A})$, and so $p \Vdash \text{VP}(\dot{A})$. \dashv

§5. Reverse Easton iterations. Having shown above that Vopěnka cardinals are preserved by small forcing, we modify the argument to show that they are moreover preserved by all generics for typical κ -length iterations. This contrasts with the situation for most strong large cardinals, which are preserved only when generics are carefully chosen to contain suitable master conditions. The key idea remains the same as in the previous section: because we do not have to preserve all of the embeddings present in the ground model, a density argument, which can be carried out without extra assumptions or preparation, will suffice.

Recall that a forcing iteration has *Easton support* if direct limits are taken at inaccessible cardinal stages and inverse limits at other limit stages. We call a forcing iteration (possibly of class length) a *reverse Easton iteration* if it has Easton support. See Cummings [11] for more on such iterations.

The precise statement that we shall prove for Vopěnka cardinals is the following.

THEOREM 15. *Let κ be a Vopěnka cardinal. Suppose $\langle P_\alpha \mid \alpha \leq \kappa \rangle$ is the reverse Easton iteration of $\langle \dot{Q}_\alpha \mid \alpha < \kappa \rangle$, where each \dot{Q}_α has cardinality less than κ , and for every $\gamma < \kappa$, there is an η_0 such that for all $\eta \geq \eta_0$,*

$$\Vdash_{P_\eta} \dot{Q}_\eta \text{ is } \gamma\text{-directed-closed}.$$

Then

$$\Vdash_{P_\kappa} \kappa \text{ is a Vopěnka cardinal}.$$

Note also that a full class-length iteration will often preserve all Vopěnka cardinals by Theorem 15 and Corollary 10, so long as the tail of the iteration from any Vopěnka cardinal κ is κ^+ -directed closed.

Since a direct limit is taken at κ , we can and will identify conditions in P_κ with conditions in $\bigcup_{\alpha < \kappa} P_\alpha$. Further, we observe that

the “smallness” requirement on the names \dot{Q}_α results in a certain amount of closure with respect to features of the forcing iteration being reflected downwards.

DEFINITION 16. *Let $\langle P_\alpha \mid \alpha \leq \kappa \rangle$ be a forcing iteration as in the statement of Theorem 15. We say that a Mahlo cardinal $\delta < \kappa$ is P_κ -reflecting if*

1. $|P_\delta|$ is at most δ , and
2. for all $\eta \geq \delta$, $\Vdash_{P_\eta} \dot{Q}_\eta$ is δ -directed-closed.

LEMMA 17. *Suppose κ is a Vopěnka cardinal, and $\langle P_\alpha \mid \alpha \leq \kappa \rangle$ is a forcing iteration as in the statement of Theorem 15. Then the set of P_κ -reflecting Mahlo cardinals is stationary in κ .*

PROOF. Vopěnka cardinals are 2-Mahlo — see Kanamori [23, Corollary 24.17]. It therefore suffices to show that the set of cardinals

$$\{\gamma < \kappa \mid \left| \bigcup_{\alpha < \gamma} P_\alpha \right| \leq \gamma \wedge \forall \eta \geq \gamma (\Vdash_{P_\eta} \dot{Q}_\eta \text{ is } \gamma\text{-directed-closed.})\}$$

is closed unbounded in κ . But this follows from a standard closure argument. \dashv

For every $\gamma < \kappa$, we shall denote by $\delta(\gamma)$ the least P_κ -reflecting Mahlo cardinal strictly greater than γ .

As in the previous section, an important part of the proof is that we can be very selective about the kind of names with which we work.

DEFINITION 18. *Let P_κ be a forcing iteration as in the statement of Theorem 15. A nice local P_κ -name for an ordinal \mathcal{L} -structure is a name σ satisfying the following requirements:*

1. σ is the canonical name for an ordinal \mathcal{L} -structure $\langle \gamma_\sigma, R^\sigma \mid R \in \mathcal{L} \rangle$ in $V[G]$ with components named by $\check{\gamma}_\sigma$ and \dot{R}^σ for $R \in \mathcal{L}$,
2. for every n -ary $R \in \mathcal{L}$, the name \dot{R}^σ is a $P_{\delta(\gamma_\sigma)}$ -name for a subset of γ_σ^n .

LEMMA 19. *For any finite language \mathcal{L} and forcing iteration P_κ as in the statement of Theorem 15, given a name \dot{A} and a condition $p \in P_\kappa$ such that*

$$p \Vdash \dot{A} \text{ is a set of ordinal } \mathcal{L}\text{-structures,}$$

there is a name \dot{B} such that $p \Vdash \dot{A} = \dot{B}$, and for every element $\langle \sigma, q \rangle$ of \dot{B} , σ is a nice local name for an ordinal \mathcal{L} -structure and $q \leq p$.

Finiteness of \mathcal{L} is a stronger requirement than is necessary, but it is convenient and suffices for our purposes.

PROOF. The proof is much like that for Lemma 13, except that we dictate that the conditions q that we use should satisfy more

stringent requirements. For each $\langle \dot{a}, r \rangle \in \dot{A}$, let

$$Q_{\dot{a}, r} = \left\{ q \in P_\kappa \mid q \leq p \wedge q \leq r \wedge \exists \gamma \left((q \Vdash \text{dom}(\dot{a}) = \gamma) \wedge \forall R \in \mathcal{L} \exists \dot{R} (\dot{R} \text{ is a } P_{\delta(\gamma)}\text{-name} \wedge (q \Vdash R^{\dot{a}} = \dot{R})) \right) \right\}$$

where $R^{\dot{a}}$ denotes the canonical name for the interpretation of R in \dot{a}_G , and as is standard practice, we abuse notation regarding for which partial order any given name is actually a name. This set $Q_{\dot{a}, r}$ will be dense for conditions below both p and r , as P_κ may be factorised as $P_{\delta(\gamma)} * \dot{P}^{[\delta(\gamma), \kappa]}$ with

$$\Vdash_{P_{\delta(\gamma)}} P^{[\delta(\gamma), \kappa]} \text{ is } \delta(\gamma)\text{-directed-closed}$$

(by for example Corollary 2.4 and Theorem 5.5 of Baumgartner [7]) and so in particular every subset of γ in the extension can be named by a $P_{\delta(\gamma)}$ -name. Constructing names $\sigma_{\dot{a}, r, q}$ from the γ and names \dot{R} from the definition of $Q_{\dot{a}, r}$, the name $\dot{B} = \{ \langle \sigma_{\dot{a}, r, q}, q \rangle \mid \langle \dot{a}, r \rangle \in \dot{A} \}$ will be as desired. \dashv

PROOF OF THEOREM 15. Let κ and P_κ be as in the statement of the theorem, and let G be P_κ -generic over V ; we shall show that κ remains Vopěnka in $V[G]$. Suppose that \dot{A} is a P_κ -name, and $p \in G$ is such that

$$p \Vdash \dot{A} \subset V_{\check{\kappa}} \wedge |\dot{A}| = \check{\kappa} \wedge \dot{A} \text{ is a set of ordinal } \mathcal{L}\text{-structures in } V_{\check{\kappa}}.$$

Let \dot{B} be as in Lemma 19. Using the Proposition 7 characterisation of Vopěnka cardinals, let $\alpha < \kappa$ be extendible below κ for \dot{B} in V . Let ξ be the least P_κ -reflecting Mahlo cardinal such that there is a $\langle \sigma, q \rangle \in \dot{B} \setminus V_\alpha$ where $q \in G \cap P_\xi$ and σ names an ordinal \mathcal{L}_{std} -structure $\langle \gamma, E, R \rangle$ with $\alpha \leq \gamma < \xi$ (whence \dot{E} and \dot{R} will be P_ξ -names). We may factorise P_κ as $P_\xi * P^{[\xi, \kappa]}$; we shall show that given $G_\xi = G \cap P_\xi$, it is dense in $P^{[\xi, \kappa]}$ to be a master condition for an embedding from σ_{G_ξ} to another element of B .

Towards that end, consider an arbitrary \dot{r} forced to be in $\dot{P}^{[\xi, \kappa]}$. We have chosen ξ such that $|P_\xi| \leq \xi$ and a direct limit is taken at κ , hence, we may take $\eta < \kappa$ large enough that

$$\Vdash_{P_\xi} \dot{r} \in \dot{P}^{[\xi, \eta]}$$

(see for example Jech [19, Lemma 21.8]) and

$$\Vdash_{P_\eta} \dot{P}^{[\eta, \kappa]} \text{ is } |G_\xi|^+\text{-directed-closed.}$$

Let $j : V_\eta \rightarrow V_\lambda$ in V be an elementary embedding witnessing the η -extendibility of α below κ for \dot{B} . In particular, this entails that $j(\alpha) > \eta$. The cardinal α is inaccessible, and so for any $q \in P_\eta$, $\text{supp}(q) \cap \alpha$ is bounded by some $\beta < \alpha$. Therefore, by elementarity and the fact that $\text{crit}(j) = \alpha$, $\text{supp}(j(q)) \cap j(\alpha)$ is bounded by β , and

$j(q) \restriction \beta = q \restriction \beta$. Note that this implies that $\text{supp}(j(q)) \cap [\xi, \eta) = \emptyset$, and we can extend $j(q)$ to $j(q) \wedge r$, where

$$(j(q) \wedge r)(\zeta) = \begin{cases} r(\zeta) & \text{if } \xi \leq \zeta < \eta \\ j(q)(\zeta) & \text{otherwise.} \end{cases}$$

Since G_ξ is a filter, j^*G_ξ is directed, so by the choice of η there is a single master condition $g \in (P^{[j(\alpha), \kappa)})^{V[G_\xi]}$ such that $g \leq j(q) \restriction [j(\alpha), \kappa)$ for all $q \in G_\xi$. Thus, the condition $g \wedge r \in P^{[\xi, \kappa)}$ extends r and forces that $j \restriction V_\xi : V_\xi \rightarrow V_{j(\xi)}$ lifts to an embedding $j' : V_\xi[G_\xi] \rightarrow V_{j(\xi)}[G_{j(\xi)}]$ defined by $j'(\dot{x}_{G_\xi}) = (j(\dot{x}))_{G_{j(\xi)}}$, which is well-defined and elementary by the definability of forcing. Note that $V_\xi[G_\xi] = V_\xi^{V[G_\xi]}$ since ξ is P_κ -reflecting, and similarly for $j(\xi)$ by elementarity.

Now consider $\langle \sigma, q \rangle \in \dot{B}$, chosen above. We have $j(\langle \sigma, q \rangle) \in \dot{B}$ by the choice of j , that is, $\langle j(\sigma), j(q) \rangle \in \dot{B}$. Since $q \in G_\xi$ by assumption, $g \wedge r \Vdash_{P[\xi, j(\xi)]} j(q) \in \dot{G}_{j(\xi)}$. By the definability of satisfaction for models and the elementarity of j' , we have

$$g \wedge r \Vdash_{P[\xi, j(\xi)]} j' \restriction \sigma_G : \sigma_G \rightarrow j'(\sigma_G) = j(\sigma)_G \text{ is elementary.}$$

Of course, $j' \restriction \sigma_G$ cannot be the identity, as the domain of σ_G is at least $\alpha = \text{crit}(j)$. Thus $g \wedge r$ extends r and forces there to be a non-trivial elementary embedding between two distinct elements of $\dot{B}_G = A$, as was required. \dashv

With Theorem 15 at our disposal, many relative consistency results become immediate by standard techniques. As an example, we list a few principles familiar from Gödel's constructible universe L .

COROLLARY 20. *If the existence of Vopěnka cardinals is consistent, then the existence of Vopěnka cardinals is also consistent with each of the following.*

1. GCH
2. $V = HOD$
3. $\diamond_{\kappa^+}^+$ holds for every infinite cardinal κ .
4. Morasses exist at every uncountable non-Vopěnka cardinal.

PROOF. There are known reverse Easton iterations of increasingly directed closed forcings to obtain each of the listed properties, such that the tail of the iteration from any Vopěnka cardinal κ is κ^+ -directed closed. For GCH, see Jensen [20], for $V = HOD$ see Brooke-Taylor [8] or Asperó and Friedman [5], for $\diamond_{\kappa^+}^+$ see Cummings, Foreman and Magidor [12, Section 12], and for morasses see Velleman [32] or [33] or Brooke-Taylor and Friedman [9]. \dashv

Part 4 of Corollary 20 raises the following question.

OPEN QUESTION. *Is it consistent, relative to the existence of a Vopěnka cardinal, to have a morass at a Vopěnka cardinal?*

Note that Vopěnka cardinals generally do not have the kind of downward reflection properties that one usually expects of strong

large cardinals, as they are not themselves weakly compact in general. Thus, it should not be surprising that Theorem 15 can be used to make properties that hold at κ fail everywhere below κ .

We also observe that in the proof of Theorem 15, the assumption of increasing directed-closure for the partial orders \dot{Q}_α , as opposed to some weaker form of closure, was necessary. In particular, using α -closed forcings, one can obtain α -Kurepa trees on inaccessible α , and α will not be ineffable in the extension — see Cummings [11, Section 6]. But there must be many ineffable cardinals below any Vopěnka cardinal, so a reverse Easton iteration of such forcings must destroy all Vopěnka cardinals.

§6. Definable Vopěnka’s Principle. We now extend the results of the previous sections to the definable class form of Vopěnka’s Principle. The forcing partial orders used in this section will correspondingly not always be sets. Rather, in Theorem 25 we shall consider class-length reverse Easton iterations of increasingly directed-closed set forcings. However, such class forcings are very well behaved; they are tame in the sense of Friedman [14], and in particular they preserve ZFC and have a definable forcing relation as for set forcing.

The first issue to address is that of names. Thanks to the definability of the forcing relation, we can have ground model “names” for classes in the extension, in the following sense.

LEMMA 21. *Let $V[G]$ be a (set- or tame class-) generic extension of V , and let A be a definable class in $V[G]$. Then there is a definable class \dot{A} in V such that for every $x \in V[G]$, $x \in A$ if and only if there is a $\langle \dot{x}, p \rangle \in \dot{A}$ such that $(\dot{x})_G = x$ and $p \in G$.*

PROOF. Suppose A is of the form

$$A = \{x \in V[G] \mid \varphi(x, z)\}$$

for some parameter $z \in V[G]$. Fix a name \dot{z} for z . Then

$$\dot{A} = \{\langle \dot{x}, p \rangle \mid p \Vdash \varphi(\dot{x}, \dot{z})\}$$

is as required. ⊢

We shall refer to such an \dot{A} as a *class name*.

We will of course need to use definable class forms of some of the properties of Vopěnka cardinals that we have used, but fortunately these are mostly provided in Solovay, Reinhardt and Kanamori [31]. Using these results we can moreover prove the definable class version of Lemma 5 without assuming $V = \text{HOD}$. We begin with the analogue of Definition 6.

DEFINITION 22. *Let A be a proper class. A cardinal $\alpha < \eta$ is η -extendible for A if there is some ζ and an elementary embedding $j : \langle V_\eta, \in, A \cap V_\eta \rangle \rightarrow \langle V_\zeta, \in, A \cap V_\zeta \rangle$ with critical point α and $j(\alpha) > \eta$. A cardinal α is A -extendible if it is η -extendible for A for all $\eta > \alpha$.*

LEMMA 23. *The following are equivalent.*

1. *Vopěnka's Principle*
2. *For every proper class A there is an A -extendible cardinal.*
3. *For every proper class A of ordinal \mathcal{L} -structures, there exist \mathcal{M} and \mathcal{N} in A of different cardinalities such that there is an elementary embedding $j : \mathcal{M} \rightarrow \mathcal{N}$.*

PROOF. (1) \Rightarrow (2) here is (1) \Rightarrow (2) of Theorem 6.9 of Solovay, Reinhardt and Kanamori [31]; briefly, structures are constructed for each α with reference to the least failure of η -extendibility of α for A , and then a Vopěnka's Principle embedding for this class of structures would yield a contradiction if there were no A -extendible α .

For (2) \Rightarrow (3), let A be as in (3), and let α be A -extendible. If $\mathcal{M} \in A$ is such that $\text{dom}(\mathcal{M}) \geq \alpha$, and j witnesses that α is $(\text{rank}(\mathcal{M}) + 1)$ -extendible for A , then $j \upharpoonright \text{dom}(\mathcal{M}) : \mathcal{M} \rightarrow j(\mathcal{M})$ is elementary, and $j(\mathcal{M}) \in A$ has cardinality $|j(\mathcal{M})| \neq |\mathcal{M}| \geq \alpha$.

Finally, for (3) \Rightarrow (1), let A be a class of \mathcal{L}_{std} -structures. Let B be the class of *all* ordinal \mathcal{L}_{std} -structures $\bar{\mathcal{M}}$ such that $\bar{\mathcal{M}}$ is isomorphic to some \mathcal{M} in A . By the Axiom of Choice, there will be elements of B isomorphic to any given element of A , but since we are not choosing representatives, we do not need to appeal to definable global choice, that is, $V = \text{HOD}$. Applying (3) to B , we get members of B with an elementary embedding between them which have different cardinalities, and thus correspond to different elements of A . We thus get an elementary embedding between distinct members of A , as required. \dashv

We are now ready to translate our results to the definable class setting. The natural Lévy–Solovay theorem for definable Vopěnka's Principle is with “set-sized” taking the place of “small”.

THEOREM 24. *Suppose Vopěnka's Principle holds in V , \mathbb{P} is a partially ordered set in V , and G is \mathbb{P} -generic over V . Then Vopěnka's Principle holds in $V[G]$.*

PROOF. The proof is as for Theorem 14. Lemma 13 is equally valid for class names \dot{A} , using no more choice than a well-order on the set of antichains of \mathbb{P} . This much choice is also sufficient for the thinning out of \dot{B}_r to a name \dot{C} , thanks to the simple form of the names in \dot{B}_r ; moreover, \dot{C} can be taken such that for different σ_0 and σ_1 appearing in \dot{C} , $|\gamma_{\sigma_0}| \neq |\gamma_{\sigma_1}|$, so that Lemma 23 will apply. The family D of structures of the proof of Theorem 14 now becomes a proper class, and the rest of the proof goes through unchanged. \dashv

We thus come to the definable form of our main theorem.

THEOREM 25. *Assume Vopěnka's Principle. Suppose*

$$P = \lim_{\rightarrow} (\langle P_\alpha \mid \alpha \in \text{Ord} \rangle)$$

is the reverse Easton iteration of $\langle \dot{Q}_\alpha \mid \alpha \in \text{Ord} \rangle$, where for every ordinal γ , there is an η_0 such that for all $\eta \geq \eta_0$,

$$\Vdash_{P_\eta} \dot{Q}_\eta \text{ is } \gamma\text{-directed closed.}$$

Then in any P -generic extension, Vopěnka's Principle holds.

PROOF. Theorem 6.6 of Solovay, Reinhardt and Kanamori [31] shows that extendible cardinals, and so in particular Mahlo cardinals, are stationary in Ord, and so the analogue of Lemma 17 goes through. Converting Lemma 19 is unproblematic. Lemma 23 above (or indeed Theorem 6.9 of [31]) gives the appropriate analogue of Proposition 7, and the rest of the proof translates smoothly. \dashv

COROLLARY 26. *If Vopěnka's Principle is consistent, then Vopěnka's Principle is also consistent with each of the following.*

1. *GCH*
2. *$V = HOD$*
3. *$\diamond_{\kappa^+}^+$ holds for every infinite cardinal κ .*
4. *Morasses exist at every uncountable cardinal.* \dashv

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